Delayed Positive Feedback Can Stabilize Oscillatory Systems

C. Abdallah, P. Dorato, and J. Benitez-Read
Department of EECE
University of New Mexico
Albuquerque, NM 87131, USA
and
R. Byrne
Sandia National Laboratories, Department 9616
Albuquerque, NM 87185

March 21, 2003

ABSTRACT

This paper expands on a method proposed in [1] for stabilizing oscillatory systems with positive, delayed feedback. The closed-loop system obtained is shown (using the Nyquist criterion) to be stable for a range of delays.

1 Introduction

The stabilization of oscillatory systems finds applications in robotics [2] and flexible structures [1]. A simple example of an oscillatory system is given by the second-order system

$$\ddot{y} + w_0^2 y = u \tag{1}$$

This class of systems can be stabilized with negative derivative feedback, i.e.

$$u(t) = -k\dot{y}(t) \ ; \quad k > 0 \tag{2}$$

The closed-loop system then becomes

$$\ddot{y} + k\dot{y} + w_0^2 y = 0 \tag{3}$$

which is obviously stable for $k > 0$. This feedback will require the differentiation of the output, or the use of an observer to estimate $\dot{y}$ from the measurement of y. This paper will present an *exact anaylsis* of a method given in [1] to stabilize this system using instead *positive delayed output feedback* only, i.e.

$$u(t) = ky(t - \tau) \tag{4}$$

In [1], the analysis of the closed-loop system was done using a first-order Padé approximation of the pure delay. In addition, no attempt was made to determine the range of allowable

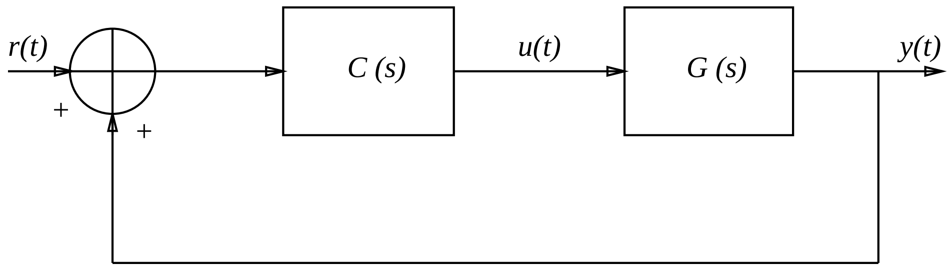

Figure 1: Block Diagram of Oscillatory System with Positive, Delay Feedback

delays in order to guarantee stability. A root locus approach was presented for such systems in [3], [4] and more recently in [5]. Note that in general, a double-integrator system described by

$$\ddot{y}(t) = u(t) \tag{5}$$

can be reduced to the oscillatory problem above by use of output-plus-delayed-output feedback of the form

$$u(t) = -w_0^2 y(t) + ky(t-\tau) \tag{6}$$

A double-integrator system will result, for example, from applying feedback-linearization to many nonlinear systems [6]. By stabilizing these systems using output feedback only, savings in sensors (tachometers) or observers are achieved. This paper will analyze the closed-loop stability of this type of system

The remaining of the paper is organized as follows. Section 2 contains the analysis of the delayed, positive-feedback control as applied to an oscillatory system. Section 3 presents examples to illustrate the value of this approach, and Section 4 contains our conclusions.

2 Analysis

Consider the plant given by

$$G(s) = \frac{1}{s^2 + w_0^2} \tag{7}$$

and the positive-feedback, time-delay compensator

$$C(s) = ke^{-s\tau} \tag{8}$$

where $k > 0$ in a simple unity-feedback loop shown in Figure 1, such that the closed-loop system is given by

$$\begin{aligned} T(s) &= \frac{G(s)C(s)}{1 - G(s)C(s)} \\ &= \frac{ke^{-s\tau}}{s^2 + w_0^2 - ke^{-s\tau}} \end{aligned} \tag{9}$$

We will study the stability of the closed-loop system by exploring the Nyquist plot of

$$-G(s)C(s) = \frac{-ke^{-s\tau}}{s^2 + w_0^2} \tag{10}$$

The Nyquist contour is assumed to be indented at the open-loop poles $\pm jw_0$ so that no poles exist in the RHP. Thus for closed-loop stability there should be no clockwise encirclements of the $(-1, 0)$ point. First, note that with $\tau = 0$, the closed-loop system is unstable because the Nyquist plot will always encircle the (-1,0) point. Consider then the case where $\tau > 0$, and note that a necessary condition for stability is that

$$k < w_0^2 \tag{11}$$

If (11) does not hold there will always be at least one clockwise encirclement. Assuming that this condition holds, let us consider the instability mechanisms by counting the number of encirclements of -1 by the polar plot of

$$-G(jw)C(jw) = \frac{-ke^{-jw\tau}}{w_0^2 - w^2} \tag{12}$$

Note that we have 3 important regions:

1. $w < w_0$
2. $w = w_0$
3. $w > w_0$

At $w = w_0$, the magnitude of the polar plot goes to infinity. This point will be studied later. Let us consider what happens to both magnitude and phase as w goes from 0 to $w_0 - \epsilon$, and then from $w_0 + \epsilon$ to ∞. The phase is given by

$$\begin{aligned} \theta(w) &= -\pi - w\tau; \quad 0 \le w < w_0 \\ &= -2\pi - w\tau; \quad w > w_0 \end{aligned} \tag{13}$$

and the magnitude by

$$\begin{aligned} \mid G(jw)C(jw) \mid &= \frac{k}{w_0^2 - w^2}; \quad 0 \le w < w_0 \\ &= \frac{k}{w^2 - w_0^2}; \quad w > w_0 \end{aligned} \tag{14}$$

Let us then find all intersections of the polar plot with the negative real axis. The intersections will take place whenever the phase is $-(2n+1)\pi,\ n = 0, 1, \cdots$. Therefore, they will take place at the frequencies w_c

$$\begin{aligned} -\pi - w_c\tau &= -(2n+1)\pi; \quad 0 \le w_c < w_0 \\ -2\pi - w_c\tau &= -(2n+1)\pi; \quad w_c > w_0 \end{aligned} \tag{15}$$

or

$$\begin{aligned} w_c\tau &= 2n\pi; \quad 0 \le w_c < w_0 \\ w_c\tau &= (2n+1)\pi; \quad w_c > w_0 \end{aligned} \tag{16}$$

In order to make sure that no encirclements of the -1 point take place, we must guarantee that the magnitude $\mid G(jw)C(jw) \mid$ evaluated at w_c is less than 1, i.e.

$$\begin{aligned} \frac{k}{w_0^2 - (4n^2\pi^2)/\tau^2} &< 1; \quad 0 \leq 2n\pi/\tau < w_0 \\ \frac{k}{(2n+1)^2\pi^2/\tau^2 - w_0^2} &< 1; \quad (2n+1)\pi/\tau > w_0 \end{aligned} \tag{17}$$

Combining both conditions we get, given that $k < w_0^2$,

$$\frac{2n\pi}{\sqrt{w_0^2 - k}} < \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} \tag{18}$$

Now, let us consider what happens at $w = w_0$. Since the magnitude is infinite at $w = w_0$, we should make sure that the phase can never be $-(2n+1)\pi$ at that frequency. In other words, we need to make sure that

$$\frac{2n\pi}{w_0} < \tau < \frac{(2n+1)\pi}{w_0} \tag{19}$$

Therefore, combining all conditions, we have the following 2 conditions

$$k < w_0^2 \tag{20}$$

$$\frac{2n\pi}{w_0} < \frac{2n\pi}{\sqrt{w_0^2 - k}} < \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} < \frac{(2n+1)\pi}{w_0} \tag{21}$$

For all $n = 0, 1, \cdots$. Note that w_0^2 can be modified if necessary by proportional feedback $-fy(t)$ in (4), i.e.

$$u(t) = -fy(t) + ky(t - \tau) \tag{22}$$

so that w_0^2 becomes

$$W_n^2 = w_0^2 + f \tag{23}$$

Also note that we can solve for the allowable region of k explicitely by finding the point of intersection of the lower and upper bounds in (21) to obtain

$$\begin{aligned} 0 &< k \leq \frac{1+4n}{1+4n+8n^2} w_0^2 \\ \frac{2n\pi}{\sqrt{w_0^2 - k}} &< \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} \end{aligned} \tag{24}$$

See the plots in Figure 2, for $w_0^2 = 1$. In particular, note that the region of stabilizing k shrinks as the delay τ gets larger. The next section presents an example of the application of this controller.

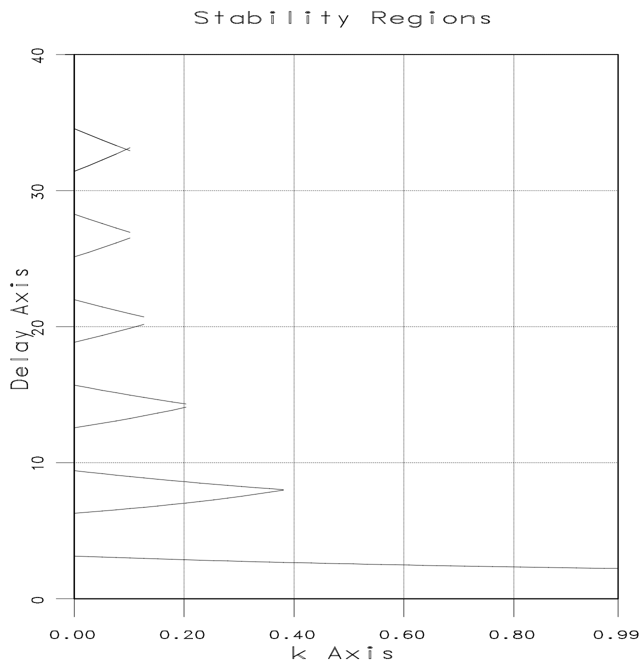

Figure 2: Stability Regions (shaded) for $w_0^2 = 1$

3 Examples

The first example will illustrate the stability and instability regions shown in Figure 2.

Example 1 Consider the open-loop system

$$G(s) = \frac{1}{s^2 + 1}$$

and let the controller be

$$u(t) = \frac{3}{13} y(t - 7.3)$$

The simulation is started at $y(0) = \dot{y}(0) = 0.1$, and is illustrated in Figure 3. Note that this example illustrates the stable region for $n = 1$. On the other hand, let

$$u(t) = \frac{6}{13} y(t - 8)$$

and if the simulation is again started at $y(0) = \dot{y}(0) = 0.1$, the trajectories in Figure 4 are obtained. These trajectories illustrate the unstable region for $n = 1$.

4 Conclusions

One normally thinks of positive feedback and pure delays as destabilizing effects in a feedback system. However for purely oscillatory systems as illustrated by the second-order system in

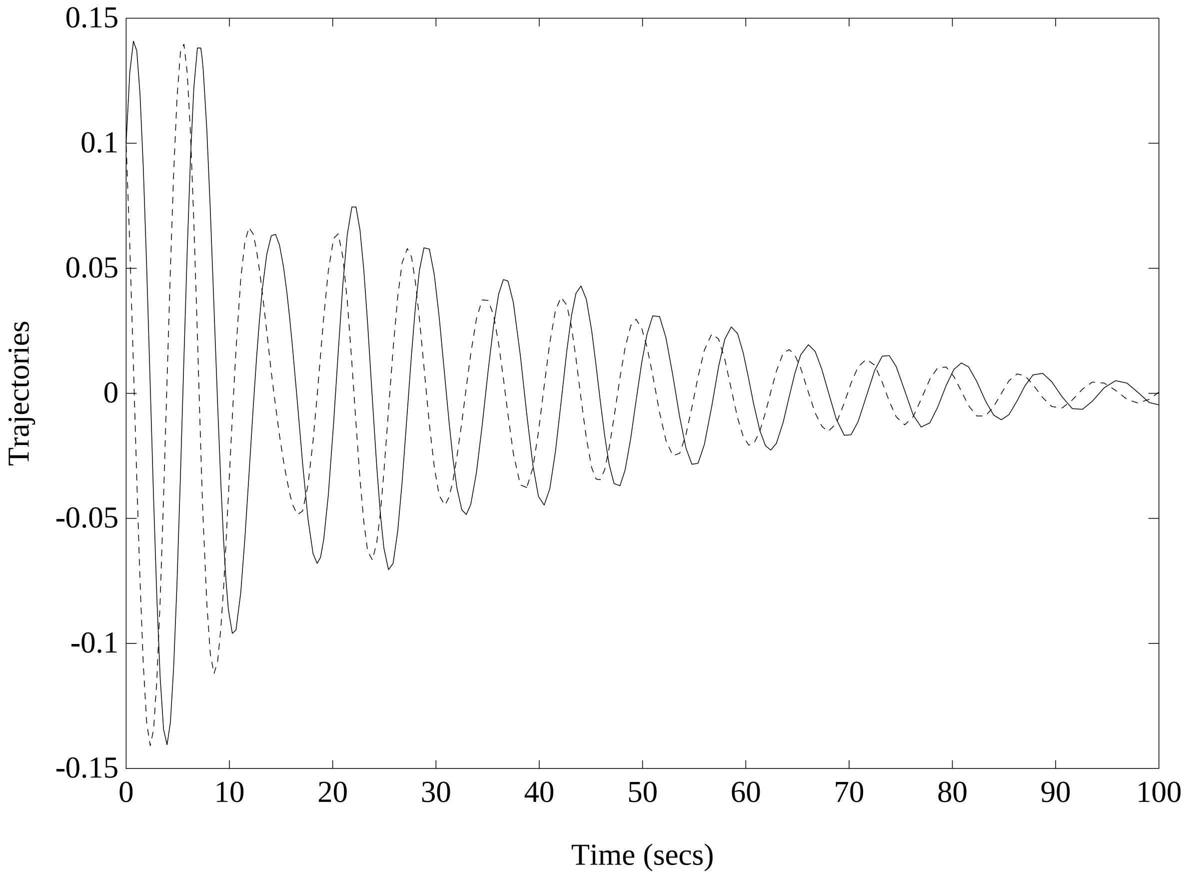

Figure 3: Stable Feedback

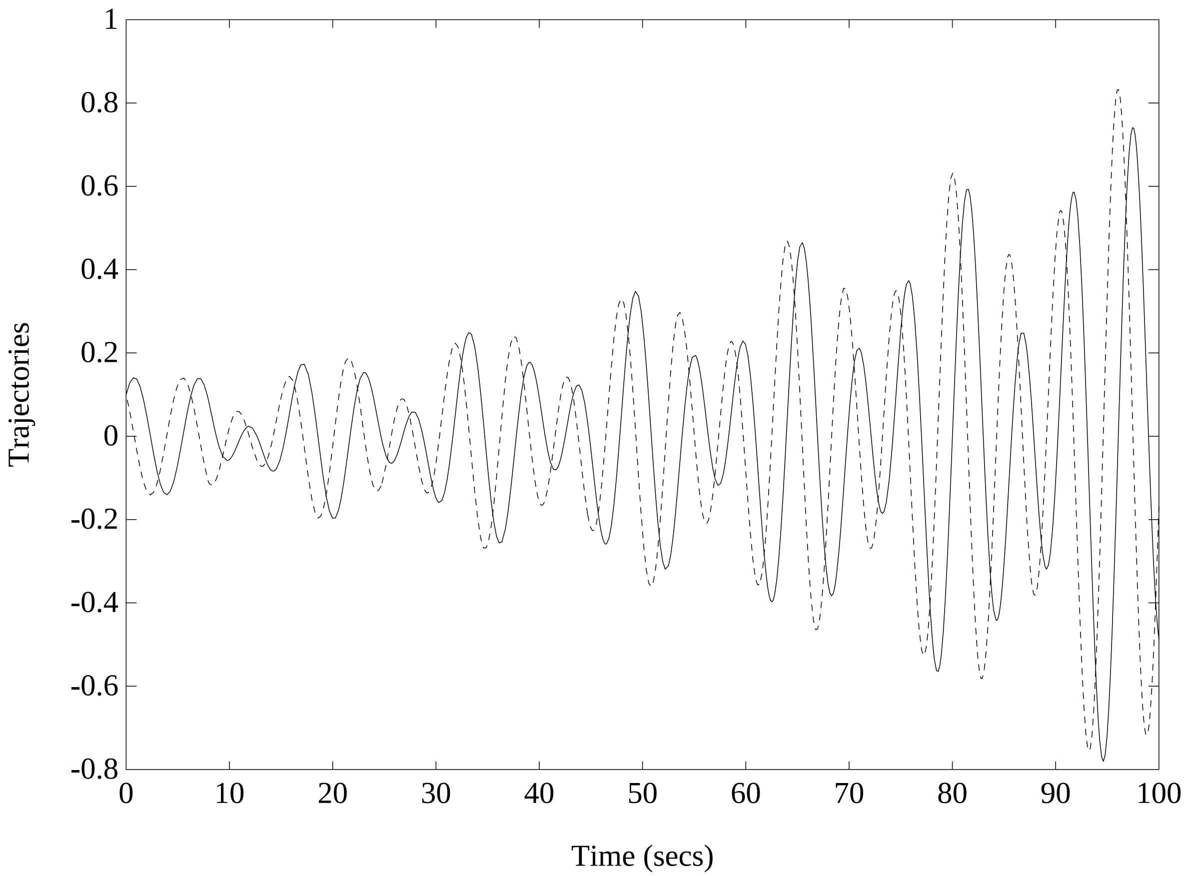

Figure 4: Unstable Feedback

this paper, this type of feedback is actually stabilizing ; and indeed since it involves only output feedback, it can result in a simpler controller.

References

[1] B. Petterson, R. Robinett, and J. Werner, "Lag-stabilized force feedback damping," *Sandia National Laboratories, SAND91-0194, UC-406*, 1991.

[2] C. Abdallah, D. Dawson, P. Dorato, and M. Jamshidi, "Survey of robust control for rigid robots," *IEEE Control Systems Mag.*, vol. 11, pp. 24–30, February 1991.

[3] T. Kurfess and M. Nagurka, "Understanding the root locus using gain plots," *IEEE Contr. Syst. Mag.*, vol. 11, pp. 37–40, August 1991.

[4] M. Nagurka and T. Kurfess, "Gain and phase margins of siso systems form modified root locus plots," *IEEE Contr. Syst. Mag.*, vol. 12, pp. 123–127, June 1992.

[5] J. Gribble, "Modified root locus plots for siso systems with time delay," *IEEE Contr. Syst. Mag.*, vol. 13, pp. 54–56, February 1993.

[6] K. Kreutz, "On manipulator control by exact linearization," *IEEE Trans. Auto. Cont.*, vol. 34, pp. 763–767, July 1989.